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More Straightforward Extraction of the Fundamental Lepton Mixing Parameters from Long-Baseline Neutrino Oscillations

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Abstract

We point out the simple reversibility between the fundamental neutrino mixing parameters in vacuum and their effective counterparts in matter. The former can therefore be expressed in terms of the latter, allowing more straightforward extraction of the genuine lepton mixing quantities from a variety of long-baseline neutrino oscillation experiments. In addition to the parametrization-independent results, we present the formulas based on the standard parametrization of the lepton flavor mixing matrix and give a typical numerical illustration.

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I. INTRODUCTION

The observation of solar and atmospheric neutrino oscillations in the Super-Kamiokande experiment has provided the most convincing evidence that neutrinos are massive and lepton flavors are mixed [1] [1]. It opens an important window for deep insight into the dynamics of fermion mass generation and flavor mixing in particle physics, and has important implications in astrophysics and cosmology.

Although some useful constraints on the lepton flavor mixing matrix can be obtained from the Super-Kamiokande measurements and other non-accelerator neutrino experiments, a precise determination of its parameters has to rely on the new generation of accelerator neutrino experiments with very long baselines [3], including possible neutrino factories [4]. The terrestrial matter effects in all long-baseline neutrino experiments must be taken into acount, since they unavoidably modify the genuine behaviors of neutrino oscillations in vacuum.

To express the pattern of neutrino oscillations in matter in the same form as that in vacuum, one may define the *effective* neutrino masses $\tilde{m}_i$ and the *effective* lepton flavor mixing matrix $\tilde{V}$ in which the terrestrial matter effects are already included. In this commonly-accepted approach, it is necessary to find out the concrete relations between the fundamental quantities of lepton mixing in vacuum (m_i and V) and their effective counterparts in matter ($\tilde{m}_i$ and $\tilde{V}$). The analytically exact formulas of $\tilde{m}_i$ and $\tilde{V}$ as functions of m_i and V have been achieved in the three-neutrino mixing scheme [5–7], but no effort has been made to derive the expressions of m_i and V in terms of $\tilde{m}_i$ and $\tilde{V}$. The latter case is equivalently interesting in phenomenology, because our physical purpose is to determine the fundamental parameters of lepton flavor mixing from the effective ones, which can directly be observed from a variety of long-baseline neutrino oscillation experiments.

This paper aims to present the analytically exact results of the fundamental quantities m_i and V in terms of the effective quantities $\tilde{m}_i$ and $\tilde{V}$. In Section 2, we first point out the interesting reversibility between (m_i, V) and $(\tilde{m}_i, \tilde{V})$, and then derive their explicit relations in a way independent of any specific parametrizations of the lepton flavor mixing matrix. In Section 3, the standard parametrization is introduced to describe V and $\tilde{V}$ in a parallel form. We calculate the mixing angles and the CP-violating phase(s) of V as functions of the relevant parameters of $\tilde{V}$, and give a typical numerical illustration. Section 4 is devoted to a brief summary.

II. REVERSIBILITY AND PARAMETRIZATION-INDEPENDENT FORMULAS

Let us concentrate on the flavor mixing between three active neutrinos and three charged leptons. The 3×3 lepton flavor mixing matrix V in vacuum is defined to link the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$:

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[1] The latest SNO measurement [2], together with the Super-Kamiokande data, has provided the first direct evidence that there is a muon- and (or) tau-neutrino component in the solar electron-neutrino flux.

$$V = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} . \tag{1}$$

A similar definition can be made for $\tilde{V}$, the effective counterpart of V in matter. If neutrinos are massive Dirac fermions, V or $\tilde{V}$ can be parametrized in terms of three rotation angles and one CP-violating phase. If neutrinos are Majorana fermions, however, two additional CP-violating phases are in general needed to fully parametrize V or $\tilde{V}$. The strength of CP or T violation in neutrino oscillations, no matter whether neutrinos are Dirac or Majorana particles, depends only upon a universal parameter J (in vacuum) or $\tilde{J}$ (in matter) [8]:

$$\begin{aligned} \mathrm{Im}\left(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}\right) &= J \sum_{\gamma,k} \left(\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\right) , \\ \mathrm{Im}\left(\tilde{V}_{\alpha i} \tilde{V}_{\beta j} \tilde{V}^*_{\alpha j} \tilde{V}^*_{\beta i}\right) &= \tilde{J} \sum_{\gamma,k} \left(\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\right) , \end{aligned} \tag{2}$$

where the Greek subscripts (α, β, γ) and the Latin subscripts (i, j, k) run over (e, μ, τ) and $(1, 2, 3)$, respectively. Of course, J and $\tilde{J}$ are two rephasing-invariant quantities.

The effective Hamiltonian responsible for the propagation of neutrinos in vacuum or in matter can be written as

$$\begin{aligned} H_\nu &= \frac{1}{2E} V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger , \\ \tilde{H}_\nu &= \frac{1}{2E} \tilde{V} \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} \tilde{V}^\dagger , \end{aligned} \tag{3}$$

where E is the neutrino beam energy; m_i and $\tilde{m}_i$ (for $i = 1, 2, 3$) denote the neutrino masses in vacuum and those in matter, respectively. The deviation of $\tilde{H}_\nu$ from H_ν results non-trivially from the charged-current contribution to the $\nu_e e^-$ forward scattering [9], when neutrinos travel through a normal material medium like the earth:

$$\Delta H_\nu \equiv \tilde{H}_\nu - H_\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{4}$$

where $a = \sqrt{2} G_{\rm F} N_e$ with N_e being the background density of electrons. Subsequently we assume a constant earth density profile (i.e., $N_e =$ constant), which is a very good approximation for all of the presently-proposed long-baseline neutrino experiments [2].

[2] A careful analysis with the method of the Fourier series expansion demonstrates that the matter profile effect is essentially irrelevant, if the baseline of neutrino oscillations is about 3000 km or shorter [10]. Therefore one may safely assume N_e to be a constant in those realistic medium- and long-baseline neutrino oscillation experiments with $L \leq 3500$ km. For $L > 3500$ km (in particular, $L \geq 5000$ km), however, details of the earth density profile have to be taken into account towards an accurate description of the features of terrestrial neutrino oscillations.

With the help of Eqs. (3) and (4), we obtain the following relations:

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} = V^\dagger \left[\tilde{V} \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} \tilde{V}^\dagger - \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] V \; ,$$

$$\begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} = \tilde{V}^\dagger \left[V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \tilde{V} \; , \tag{5}$$

where $A = 2aE$ depending linearly on the neutrino beam energy E. This result implies that the expression of $\tilde{m}_i$ as a function of A, m_j and V_{ej} is reversible with that of m_i, which must behave via the same function of variables A, $\tilde{m}_j$ and $\tilde{V}_{ej}$; namely,

$$\begin{aligned} m_i &= f_i(\tilde{m}_j, \; |\tilde{V}_{ej}|, \; -A) \; , \\ \tilde{m}_i &= f_i(m_j, \; |V_{ej}|, \; +A) \; . \end{aligned} \tag{6}$$

As m_i is real and positive, it depends only upon the absolute values of $\tilde{V}_{\alpha j}$. The specific texture of ΔH_ν assures that only the matrix elements $|\tilde{V}_{ej}|$, which are always associated with the matter parameter A, can affect m_i. The similar argument is valid for $\tilde{m}_i$. Analogously, there exists the simple reversibility between the expressions of $V_{\alpha i}$ and $\tilde{V}_{\alpha i}$:

$$\begin{aligned} V_{\alpha i} &= F_{\alpha i}(\tilde{m}_j, \; \tilde{V}_{\beta j}, \; -A) \; , \\ \tilde{V}_{\alpha i} &= F_{\alpha i}(m_j, \; V_{\beta j}, \; +A) \; . \end{aligned} \tag{7}$$

We observe that the reversible relations in Eqs. (6) and (7) coincide formally with the sum rules of neutrino mass and mixing parameters obtained in Ref. [11]:

$$\sum_{i=1}^{3} \left(m_i^2 V_{\alpha i} V_{\beta i}^* \right) \; = \; \sum_{i=1}^{3} \left(\tilde{m}_i^2 \tilde{V}_{\alpha i} \tilde{V}_{\beta i}^* \right) \; , \tag{8}$$

where $\alpha \neq \beta$ running over (e, μ, τ). In the limit $A = 0$, $\tilde{m}_i = m_i$ and $\tilde{V}_{\alpha i} = V_{\alpha i}$ must hold. Hence the functions f_i and $F_{\alpha i}$ might have transparent dependence on A. To find out the explicit forms of f_i and $F_{\alpha i}$ is simply an algebraic exercise in the three-neutrino mixing scheme [5–7].

Using Eqs. (6) and (7), one may obtain the formulas of m_i and $V_{\alpha i}$ straightforwardly from those of $\tilde{m}_i$ and $\tilde{V}_{\alpha i}$, which have been presented in Ref. [7] in a parametrization-independent way. We then arrive at

$$\begin{aligned} m_1^2 &= \tilde{m}_1^2 + \frac{1}{3}\tilde{x} - \frac{1}{3}\sqrt{\tilde{x}^2 - 3\tilde{y}} \left[\tilde{z} + \sqrt{3\left(1 - \tilde{z}^2\right)} \right] \; , \\ m_2^2 &= \tilde{m}_1^2 + \frac{1}{3}\tilde{x} - \frac{1}{3}\sqrt{\tilde{x}^2 - 3\tilde{y}} \left[\tilde{z} - \sqrt{3\left(1 - \tilde{z}^2\right)} \right] \; , \\ m_3^2 &= \tilde{m}_1^2 + \frac{1}{3}\tilde{x} + \frac{2}{3}\tilde{z}\sqrt{\tilde{x}^2 - 3\tilde{y}} \; , \end{aligned} \tag{9}$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are given by

$$
\begin{aligned}
\tilde{x} &= \Delta \tilde{m}^2_{21} + \Delta \tilde{m}^2_{31} - A \; , \\
\tilde{y} &= \Delta \tilde{m}^2_{21} \Delta \tilde{m}^2_{31} - A \left[\Delta \tilde{m}^2_{21} \left(1 - |\tilde{V}_{e2}|^2 \right) + \Delta \tilde{m}^2_{31} \left(1 - |\tilde{V}_{e3}|^2 \right) \right] \; , \\
\tilde{z} &= \cos \left[\frac{1}{3} \arccos \frac{2\tilde{x}^3 - 9\tilde{x}\tilde{y} - 27A\Delta \tilde{m}^2_{21} \Delta \tilde{m}^2_{31} |\tilde{V}_{e1}|^2}{2\left(\tilde{x}^2 - 3\tilde{y}\right)^{3/2}} \right]
\end{aligned}
\tag{10}
$$

with $\Delta \tilde{m}^2_{21} \equiv \tilde{m}^2_2 - \tilde{m}^2_1$ and $\Delta \tilde{m}^2_{31} \equiv \tilde{m}^2_3 - \tilde{m}^2_1$. Furthermore, V reads as follows:

$$
V_{\alpha i} \; = \; \frac{\tilde{N}_i}{\tilde{D}_i} \tilde{V}_{\alpha i} \; - \; \frac{A}{\tilde{D}_i} \left[\left(m^2_i - \tilde{m}^2_j \right) \tilde{V}_{\alpha k} \tilde{V}^*_{ek} + \left(m^2_i - \tilde{m}^2_k \right) \tilde{V}_{\alpha j} \tilde{V}^*_{ej} \right] \tilde{V}_{ei} \; , \tag{11}
$$

where α runs over (e, μ, τ); i, j or k runs over $(1, 2, 3)$ with $i \neq j \neq k$; and [3]

$$
\begin{aligned}
\tilde{N}_i &= \left(m^2_i - \tilde{m}^2_j \right) \left(m^2_i - \tilde{m}^2_k \right) + A \left[\left(m^2_i - \tilde{m}^2_j \right) |\tilde{V}_{ek}|^2 + \left(m^2_i - \tilde{m}^2_k \right) |\tilde{V}_{ej}|^2 \right] \; , \\
\tilde{D}^2_i &= \tilde{N}^2_i + A^2 |\tilde{V}_{ei}|^2 \left[\left(m^2_i - \tilde{m}^2_j \right)^2 |\tilde{V}_{ek}|^2 + \left(m^2_i - \tilde{m}^2_k \right)^2 |\tilde{V}_{ej}|^2 \right] \; .
\end{aligned}
\tag{12}
$$

Obviously $A = 0$ leads to $V = \tilde{V}$. The exact and compact formulas obtained above show clearly how the flavor mixing matrix in vacuum is connected to that in matter. Some instructive analytical approximations can be made for Eq. (11), once the hierarchy of effective neutrino masses and that of effective flavor mixing matrix elements are experimentally known.

With the help of Eq. (11), one may calculate the universal CP-violating parameter J in vacuum. Indeed it is easier to derive the relationship between J and $\tilde{J}$ by use of either the sum rules in Eq. (8) [11] or the equality between the determinants of lepton mass matrices in matter and in vacuum [12]. The result is

$$
J \; = \; \tilde{J} \; \frac{\Delta \tilde{m}^2_{21}}{\Delta m^2_{21}} \cdot \frac{\Delta \tilde{m}^2_{31}}{\Delta m^2_{31}} \cdot \frac{\Delta \tilde{m}^2_{32}}{\Delta m^2_{32}} \quad , \tag{13}
$$

where the neutrino mass-squared differences in vacuum can be read off from Eq. (9).

Note that the afore-obtained results are valid only for neutrinos propagating in vacuum and interacting with matter. As for antineutrinos, the respective formulas for m^2_i, V and J can straightforwardly be obtained from Eqs. (9) – (13) through the replacements $\tilde{V} \Longrightarrow \tilde{V}^*$ and $A \Longrightarrow -A$.

III. STANDARD PARAMETRIZATION AND NUMERICAL ILLUSTRATION

It is well known that the 3×3 lepton flavor mixing matrix can be parametrized in terms of three rotation angles $(\theta_1, \theta_2, \theta_3)$ and three phase angles (δ, ρ, σ), if neutrinos are Majorana particles. There are nine distinct parametrizations of this nature [13], but only one of them

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[3] In obtaining Eqs. (11) and (12), we have required that $\tilde{N}_i$ and $\tilde{D}_i$ have the same sign. Therefore $A = 0$ leads to $\tilde{N}_i / \tilde{D}_i = 1$ and $V_{\alpha i} = \tilde{V}_{\alpha i}$.

is particularly convenient in the analyses of experimental data on the neutrinoless double beta decay and neutrino oscillations. This "standard" parametrization reads in vacuum as

$$V = \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 \\ -c_1 s_2 s_3 - s_1 c_2 e^{-i\delta} & -s_1 s_2 s_3 + c_1 c_2 e^{-i\delta} & s_2 c_3 \\ -c_1 c_2 s_3 + s_1 s_2 e^{-i\delta} & -s_1 c_2 s_3 - c_1 s_2 e^{-i\delta} & c_2 c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\rho} & 0 \\ 0 & 0 & e^{i\sigma} \end{pmatrix} \tag{14}$$

with $s_i \equiv \sin\theta_i$ and $c_i \equiv \cos\theta_i$ (for $i = 1, 2, 3$). Without loss of generality, the three mixing angles $(\theta_1, \theta_2, \theta_3)$ can all be arranged to lie in the first quadrant. Arbitrary values between $-\pi$ and $+\pi$ are allowed for the CP-violating phases δ, ρ and σ. Note that the location of the Dirac-type phase δ in V is different from that advocated by the Particle Data Group in Ref. [14]. The advantage of our present phase assignment is that δ itself does not appear in the effective Majorana mass term of the neutrinoless double beta decay [13]. Note also that normal neutrino oscillations are completely insensitive to the Majorana-type phases ρ and σ, therefore the magnitudes of these two parameters keep unchanged even when neutrinos propagate in matter [15]. On the other hand, the other four parameters $\theta_1, \theta_2, \theta_3$ and δ may be contaminated by the terrestrial matter effects in realistic long-baseline neutrino oscillation experiments. In analogy to Eq. (14), the effective (matter-corrected) lepton flavor mixing matrix can be parametrized as follows:

$$\tilde{V} = \begin{pmatrix} \tilde{c}_1 \tilde{c}_3 & \tilde{s}_1 \tilde{c}_3 & \tilde{s}_3 \\ -\tilde{c}_1 \tilde{s}_2 \tilde{s}_3 - \tilde{s}_1 \tilde{c}_2 e^{-i\tilde{\delta}} & -\tilde{s}_1 \tilde{s}_2 \tilde{s}_3 + \tilde{c}_1 \tilde{c}_2 e^{-i\tilde{\delta}} & \tilde{s}_2 \tilde{c}_3 \\ -\tilde{c}_1 \tilde{c}_2 \tilde{s}_3 + \tilde{s}_1 \tilde{s}_2 e^{-i\tilde{\delta}} & -\tilde{s}_1 \tilde{c}_2 \tilde{s}_3 - \tilde{c}_1 \tilde{s}_2 e^{-i\tilde{\delta}} & \tilde{c}_2 \tilde{c}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\rho} & 0 \\ 0 & 0 & e^{i\sigma} \end{pmatrix} \tag{15}$$

with $\tilde{s}_i \equiv \sin\tilde{\theta}_i$ and $\tilde{c}_i \equiv \cos\tilde{\theta}_i$ (for $i = 1, 2, 3$). The universal CP-violating parameters J and $\tilde{J}$ defined in Eq. (2) depend respectively upon the Dirac-type phases δ and $\tilde{\delta}$; i.e.,

$$\begin{aligned} J &= s_1 c_1 s_2 c_2 s_3 c_3^2 \sin\delta \ , \\ \tilde{J} &= \tilde{s}_1 \tilde{c}_1 \tilde{s}_2 \tilde{c}_2 \tilde{s}_3 \tilde{c}_3^2 \sin\tilde{\delta} \ . \end{aligned} \tag{16}$$

The CP- and T-violating asymmetries of neutrino oscillations are proportional to J in vacuum and $\tilde{J}$ in matter.

Now let us proceed to figure out the analytical expressions of the fundamental parameters $(\theta_1, \theta_2, \theta_3, \delta)$ in terms of the effective parameters $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\delta})$, which can directly be measured from a variety of long-baseline neutrino oscillation experiments. To do so, we simply apply the standard parametrization of $\tilde{V}$ to the parametrization-independent formula in Eq. (11). Then the matrix elements of V read explicitly as

$$V_{\alpha i} = \frac{\tilde{N}_i}{\tilde{D}_i} \tilde{V}_{\alpha i} - \frac{A}{\tilde{D}_i} \sum_{k=1}^{3} \left(\tilde{T}_{\alpha k} P_{ki} \right) \ , \tag{17}$$

where $\tilde{N}_i$ and $\tilde{D}_i$ have been given in Eq. (12), $P \equiv \mathrm{Diag}\{1, e^{i\rho}, e^{i\sigma}\}$ denotes the diagonal Majorana phase matrix, and the matter-associated quantities $\tilde{T}_{\alpha k}$ are given as

$$\begin{aligned} \tilde{T}_{e1} &= +\tilde{c}_1 \tilde{c}_3 \left[\left(m_1^2 - \tilde{m}_2^2 \right) \tilde{s}_3^2 + \left(m_1^2 - \tilde{m}_3^2 \right) \tilde{s}_1^2 \tilde{c}_3^2 \right] \ , \\ \tilde{T}_{e2} &= +\tilde{s}_1 \tilde{c}_3 \left[\left(m_2^2 - \tilde{m}_1^2 \right) \tilde{s}_3^2 + \left(m_2^2 - \tilde{m}_3^2 \right) \tilde{c}_1^2 \tilde{c}_3^2 \right] \ , \end{aligned}$$

$$
\begin{aligned}
\tilde{T}_{e3} &= +\tilde{s}_3\tilde{c}_3^2\left[\left(m_3^2-\tilde{m}_1^2\right)\tilde{s}_1^2+\left(m_3^2-\tilde{m}_2^2\right)\tilde{c}_1^2\right] \ , \\
\tilde{T}_{\mu 1} &= +\tilde{c}_1\tilde{c}_3^2\left[\left(m_1^2-\tilde{m}_2^2\right)\tilde{s}_2\tilde{s}_3-\left(m_1^2-\tilde{m}_3^2\right)\left(\tilde{s}_1^2\tilde{s}_2\tilde{s}_3-\tilde{s}_1\tilde{c}_1\tilde{c}_2e^{-i\tilde{\delta}}\right)\right] \ , \\
\tilde{T}_{\mu 2} &= +\tilde{s}_1\tilde{c}_3^2\left[\left(m_2^2-\tilde{m}_1^2\right)\tilde{s}_2\tilde{s}_3-\left(m_2^2-\tilde{m}_3^2\right)\left(\tilde{c}_1^2\tilde{s}_2\tilde{s}_3+\tilde{s}_1\tilde{c}_1\tilde{c}_2e^{-i\tilde{\delta}}\right)\right] \ , \\
\tilde{T}_{\mu 3} &= -\tilde{s}_3\tilde{c}_3\left[\left(m_3^2-\tilde{m}_1^2\right)\tilde{s}_1^2\tilde{s}_2\tilde{s}_3+\left(m_3^2-\tilde{m}_2^2\right)\tilde{c}_1^2\tilde{s}_2\tilde{s}_3-\Delta\tilde{m}_{21}^2\tilde{s}_1\tilde{c}_1\tilde{c}_2e^{-i\tilde{\delta}}\right] \ , \\
\tilde{T}_{\tau 1} &= +\tilde{c}_1\tilde{c}_3^2\left[\left(m_1^2-\tilde{m}_2^2\right)\tilde{c}_2\tilde{s}_3-\left(m_1^2-\tilde{m}_3^2\right)\left(\tilde{s}_1^2\tilde{c}_2\tilde{s}_3+\tilde{s}_1\tilde{c}_1\tilde{s}_2e^{-i\tilde{\delta}}\right)\right] \ , \\
\tilde{T}_{\tau 2} &= +\tilde{s}_1\tilde{c}_3^2\left[\left(m_2^2-\tilde{m}_1^2\right)\tilde{c}_2\tilde{s}_3-\left(m_2^2-\tilde{m}_3^2\right)\left(\tilde{c}_1^2\tilde{c}_2\tilde{s}_3-\tilde{s}_1\tilde{c}_1\tilde{s}_2e^{-i\tilde{\delta}}\right)\right] \ , \\
\tilde{T}_{\tau 3} &= -\tilde{s}_3\tilde{c}_3\left[\left(m_3^2-\tilde{m}_1^2\right)\tilde{s}_1^2\tilde{c}_2\tilde{s}_3+\left(m_3^2-\tilde{m}_2^2\right)\tilde{c}_1^2\tilde{c}_2\tilde{s}_3-\Delta\tilde{m}_{21}^2\tilde{s}_1\tilde{c}_1\tilde{s}_2e^{-i\tilde{\delta}}\right] \ .
\end{aligned}
\tag{18}
$$

It should be noted that the matrix elements $V_{\mu 3}$ and $V_{\tau 3}$ in Eq. (17) are complex and dependent on the effective CP-violating phase $\tilde{\delta}$, as one can see from $\tilde{T}_{\mu 3}$ and $\tilde{T}_{\tau 3}$ in Eq. (18). Hence a proper redefinition of the phases for muon and tau fields is needed, in order to make $V_{\mu 3}$ and $V_{\tau 3}$ real in the course of linking $\tilde{V}$ in Eq. (15) to V in Eq. (14). The instructive relations between the effective mixing angles in matter $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ and the fundamental mixing angles in vacuum $(\theta_1, \theta_2, \theta_3)$ are found to be

$$
\begin{aligned}
\frac{\tan\theta_1}{\tan\tilde{\theta}_1} &= \frac{\tilde{N}_2 - A\left[\left(m_2^2-\tilde{m}_1^2\right)\tilde{s}_3^2+\left(m_2^2-\tilde{m}_3^2\right)\tilde{c}_1^2\tilde{c}_3^2\right]}{\tilde{N}_1 - A\left[\left(m_1^2-\tilde{m}_2^2\right)\tilde{s}_3^2+\left(m_1^2-\tilde{m}_3^2\right)\tilde{s}_1^2\tilde{c}_3^2\right]}\cdot\frac{\tilde{D}_1}{\tilde{D}_2} \ , \\
\frac{\tan\theta_2}{\tan\tilde{\theta}_2} &= 1-\frac{A\Delta\tilde{m}_{21}^2\tilde{s}_1\tilde{c}_1\tilde{s}_3\cos\tilde{\delta}/\left(\tilde{s}_2\tilde{c}_2\right)}{\tilde{N}_3 + A\left[\left(m_3^2-\tilde{m}_1^2\right)\tilde{s}_1^2+\left(m_3^2-\tilde{m}_2^2\right)\tilde{c}_1^2\right]\tilde{s}_3^2} \ , \\
\frac{\sin\theta_3}{\sin\tilde{\theta}_3} &= \frac{\tilde{N}_3}{\tilde{D}_3} - \frac{A}{\tilde{D}_3}\left[\left(m_3^2-\tilde{m}_1^2\right)\tilde{s}_1^2+\left(m_3^2-\tilde{m}_2^2\right)\tilde{c}_1^2\right]\tilde{c}_3^2 \ .
\end{aligned}
\tag{19}
$$

Note that we have only presented the next-to-leading order expression for $\tan\theta_2/\tan\tilde{\theta}_2$ in Eq. (19), since the exact result is too complicated to be instructive. The former works to a high degree of accuracy, and it is identical to the exact result provided that $\tilde{\theta}_2 = \pi/4$ and $\tilde{\delta} = \pm\pi/2$ hold. Once the relations between $\tilde{\theta}_i$ and θ_i (for $i = 1, 2, 3$) are fixed, one can derive the relation between the effective CP-violating phase $\tilde{\delta}$ in matter and the genuine CP-violating phase δ in vacuum by use of Eq. (16) as well as the relationship between $\tilde{J}$ and J in Eq. (13). We obtain

$$
\frac{\sin\delta}{\sin\tilde{\delta}} = \frac{\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2\tilde{s}_3\tilde{c}_3^2}{s_1c_1s_2c_2s_3c_3^2}\cdot\frac{\Delta\tilde{m}_{21}^2}{\Delta m_{21}^2}\cdot\frac{\Delta\tilde{m}_{31}^2}{\Delta m_{31}^2}\cdot\frac{\Delta\tilde{m}_{32}^2}{\Delta m_{32}^2} \ . \tag{20}
$$

Of course, $\tilde{\delta} = \delta$ holds in the limit $A = 0$. The formulas obtained in Eqs. (19) and (20) are very useful for the purpose of extracting the fundamental parameters of lepton flavor mixing from the matter-corrected ones, which can be measured from various long- and medium-baseline neutrino oscillation experiments in the near future.

Note again that the afore-obtained results are valid only for neutrinos interacting with matter. As for antineutrinos, the corresponding expressions of θ_1, θ_2, θ_3 and δ can straightforwardly be achieved from Eqs. (19) and (20) through the replacements $\tilde{\delta} \Longrightarrow -\tilde{\delta}$ and $A \Longrightarrow -A$.

It is worthwhile at this point to remark two advantages of the results presented above over those obtained by Zaglauer and Schwarzer in Ref. [6]. First, our relations clearly show the proportionality between the sine or tangent functions of $(\theta_1, \theta_2, \theta_3, \delta)$ and $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\delta})$. Second, our relations make the dependence of $\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_3$ and $\tilde{\delta}$ on the matter parameter A more transparent. Therefore our results are expected to be more useful for the analytical study of terrestrial matter effects on lepton flavor mixing and CP violation.

For numerical illustration, we assume the matter density of the earth's crust to be constant. Then $A \approx 2.28 \cdot 10^{-4}\ \text{eV}^2 E/[\text{GeV}]$ is a good approximation [16], where E is the neutrino beam energy. We typically take $E = 5$ GeV. The input values of the effective neutrino mixing parameters in matter are listed on the left side of Table 1. With the help of Eqs. (9) and (10) as well as Eqs. (19) and (20), one may calculate the fundamental neutrino mixing parameters in vacuum. We list our numerical results on the right side of Table 1 for both the case of neutrinos $(+A)$ and that of antineutrinos $(-A)$. One can see that the input values of $(\Delta\tilde{m}^2_{21}, \Delta\tilde{m}^2_{31})$ and $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\delta})$ are rather *ad hoc*. The reason is simply that we contrive to get the *appropriate* sizes of $(\Delta m^2_{21}, \Delta m^2_{31})$ and $(\theta_1, \theta_2, \theta_3, \delta)$, which are compatible with the present Super-Kamiokande data on solar and atmospheric neutrino oscillations [1]. Indeed we observe from Table 1 that $\Delta m^2_{\text{sun}} \approx \Delta m^2_{21} \approx 4.62 \cdot 10^{-5}\ \text{eV}^2$ (or $5.05 \cdot 10^{-5}\ \text{eV}^2$) and $\sin^2 2\theta_{\text{sun}} \approx \sin^2 2\theta_1 \approx 0.75$ (or 0.90) do agree with the large-angle MSW solution to the solar neutrino problem, and $\Delta m^2_{\text{atm}} \approx \Delta m^2_{31} \approx 3.00 \cdot 10^{-3}\ \text{eV}^2$ (or $2.89 \cdot 10^{-3}\ \text{eV}^2$) together with $\sin^2 2\theta_{\text{atm}} \approx \sin^2 2\theta_2 \approx 0.99$ (or 0.97) is consistent with the atmospheric neutrino oscillation data. Moreover, $\sin^2 2\theta_{\text{CHOOZ}} \approx \sin^2 2\theta_3 \approx 0.03 < 0.1$ coincides with the CHOOZ and Palo Verde reactor experiments [17]. Note that $\Delta\tilde{m}^2_{21} \sim A \gg \Delta m^2_{21}$ holds for the given value of E, hence there appears a substantial difference between θ_1 and $\tilde{\theta}_1$. On the other hand, $\theta_2 \approx \tilde{\theta}_2$ and $\delta \approx \tilde{\delta}$ hold to an excellent degree of accuracy. The tiny difference between θ_2 and $\tilde{\theta}_2$ in our numerical illustration can easily be understood: the special input $\tilde{\delta} = \pm 90^\circ$ assures the vanishing of the term proportional to $A\cos\tilde{\delta}$ on the right-hand side of the expression for $\tan\theta_2/\tan\tilde{\theta}_2$ in Eq. (19), leading straightforwardly to $\theta_2 = \tilde{\theta}_2$ in the next-to-leading order approximation. In general, $\theta_2 \approx \tilde{\theta}_2$ remains a good approximation, if the condition $|\Delta m^2_{21}| \ll |\Delta m^2_{31}|$ is satisfied [18]. One may numerically check that $\delta \approx \tilde{\delta}$ is generally valid as well, provided that $|\Delta m^2_{21}| \ll |\Delta m^2_{31}|$ holds. Analytically, however, we have not found a transparent way to show the approximate equality between δ and $\tilde{\delta}$. Some more effort is obviously desirable, in order to understand why the CP-violating phase δ is in most cases insensitive to the terrestrial matter effects [19].

Our numerical results illustrate that the fundamental neutrino mixing parameters can straightforwardly be extracted from their matter-corrected counterparts, once the latter are measured from the future long-baseline neutrino oscillation experiments. In particular, $|\Delta m^2_{21}| \ll |\Delta m^2_{31}|$ assures that $\tilde{\theta}_2 \approx \theta_2$ and $\tilde{\delta} \approx \delta$ hold in the standard parametrization of lepton flavor mixing.

IV. SUMMARY

We have pointed out that there exists the reversibility between the fundamental neutrino mixing parameters in vacuum and their effective counterparts in matter. The former can therefore be expressed in terms of the latter, allowing more straightforward extraction of

the genuine lepton mixing quantities from a variety of long-baseline neutrino oscillation experiments. Besides the parametrization-independent formalism, we have presented the formulas based on the standard parametrization of the neutrino mixing matrix and given a typical numerical illustration. Our results are expected to be a useful addition to the phenomenology of lepton flavor mixing and neutrino oscillations.

REFERENCES

[1] Super-Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett. **81**, 1562 (1998); *ibid.* **81**, 4279 (1998); http://www-sk.icrr.u-tokyo.ac.jp/dpc/sk/.

[2] SNO Collaboration, Q.R. Ahmad *et al.*, nucl-ex/0106015.

[3] P. Fisher, B. Kayser, and K.S. McFarland, Ann. Rev. Nucl. Part. Sci. **49**, 481 (1999); and references therein.

[4] B. Autin *et al.*, CERN 99-02 (1999); D. Ayres *et al.*, physics/9911009; C. Albright *et al.*, hep-ex/0008064; and references therein.

[5] V. Barger, K. Whisnant, S. Pakvasa, and R.J. Phillips, Phys. Rev. D **22**, 2718 (1980).

[6] H.W. Zaglauer and K.H. Schwarzer, Z. Phys. C **40**, 273 (1988).

[7] Z.Z. Xing, Phys. Lett. B **487**, 327 (2000).

[8] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[9] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).

[10] T. Ota and J. Sato, Phys. Rev. D **63**, 093004 (2001); and references therein.

[11] Z.Z. Xing, Phys. Rev. D **64**, 033005 (2001).

[12] P.F. Harrison and W.G. Scott, Phys. Lett. B **476**, 349 (2000); Z.Z. Xing, Phys. Rev. D **63**, 073012 (2001).

[13] H. Fritzsch and Z.Z. Xing, hep-ph/0103242.

[14] Particle Data Group, D.E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).

[15] P. Langacker, S.T. Petcov, G. Steigman, and S. Toshev, Nucl. Phys. B **282**, 589 (1987); H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000).

[16] A. Dziewonski, *Earth Structure*, in "The Encyclopedia of Solid Earth Geophysics", edited by D.E. James (Van Nostrand Reinhold, New York, 1989), p. 331.

[17] CHOOZ Collaboration, M. Apollonio *et al.*, Phys. Lett. B **420**, 397 (1998); Palo Verde Collaboration, F. Boehm *et al.*, Phys. Rev. Lett. **84**, 3764 (2000).

[18] J. Pantaleone, Phys. Rev. D **49**, 2152 (1994); V. Barger, S. Geer, R. Raja, and K. Whisnant, Phys. Rev. D **62**, 013004 (2000).

[19] Z.Z. Xing, in preparation.

TABLES

TABLE I. Numerical illustration of extracting the fundamental neutrino mixing parameters in vacuum from their effective counterparts in matter, where $A = 2.28 \cdot 10^{-4}$ eV$^2 E/$[GeV] with $E = 5$ GeV has typically been used.

Neutrinos ($+A$)	
Input Parameters	Output Parameters
$\Delta\tilde{m}^2_{21} = 1.15 \cdot 10^{-3}$ eV^2	$\Delta m^2_{21} = 4.62 \cdot 10^{-5}$ eV^2
$\Delta\tilde{m}^2_{31} = 3.00 \cdot 10^{-3}$ eV^2	$\Delta m^2_{31} = 3.00 \cdot 10^{-3}$ eV^2
$\tilde{\theta}_1 = 89°$	$\theta_1 = 60°$
$\tilde{\theta}_2 = 42°$	$\theta_2 = 42°$
$\tilde{\theta}_3 = 8°$	$\theta_3 = 5°$
$\tilde{\delta} = 90°$	$\delta = 90°$
Antineutrinos ($-A$)	
Input Parameters	Output Parameters
$\Delta\tilde{m}^2_{21} = 1.15 \cdot 10^{-3}$ eV^2	$\Delta m^2_{21} = 5.05 \cdot 10^{-5}$ eV^2
$\Delta\tilde{m}^2_{31} = 4.00 \cdot 10^{-3}$ eV^2	$\Delta m^2_{31} = 2.89 \cdot 10^{-3}$ eV^2
$\tilde{\theta}_1 = 1.2°$	$\theta_1 = 36°$
$\tilde{\theta}_2 = 40°$	$\theta_2 = 40°$
$\tilde{\theta}_3 = 3.6°$	$\theta_3 = 5°$
$\tilde{\delta} = -90°$	$\delta = -90°$